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On Classification of Finite-Dimensional Superbialgebras and Hopf Superalgebras

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Abstract. The purpose of this paper is to investigate finite-dimensional superbialgebras and Hopf superalgebras. We study connected superbialgebras and provide a classification of non-trivial superbialgebras and Hopf superalgebras in dimension n with $n \leq 4$.

Key words: superalgebra; superbialgebra; Hopf superalgebra; classification

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Introduction

Hopf superalgebras are a generalization of the supergroup notion. In physics, infinite-dimensional Hopf superalgebras based on Lie superalgebras turn out to be related to the integrable S-matrix of the AdS/CFT correspondence. For a large class of finite-dimensional Lie superalgebras (including the classical simple ones) a Lie supergroup associated to the algebra is defined by fixing the Hopf superalgebra of functions on the supergroup [34]. This framework seems to be interesting for studying supermanifolds and supersymmetry. Majid showed that the theory of Lie superalgebras and Hopf superalgebras can be reduced to the classical case using the bosonization by $\mathbb{K}[\mathbb{Z}/2\mathbb{Z}]$, where $\mathbb{K}$ is an algebraically closed field [24]. Recently some classes of Hopf superalgebras were investigated like for example pointed Hopf superalgebras and quasitriangular Hopf superalgebras [3, 4, 18], see also [13, 23]. So far very few is known about general classification of superbialgebras and Hopf superalgebras. Nevertheless, classification of finite-dimensional Hopf algebras is known for small dimensions and structure of many classes are deeply studied, see for example [2, 4, 5, 9, 10, 15, 16, 28, 29, 31, 32, 33, 36, 37, 38, 39]. For general theory of Hopf algebras, we refer to [1, 19, 21, 25, 30, 35], and for applications see for example [12, 20, 22, 27].

In this paper, we discuss properties of n-dimensional superbialgebras and provide a classification of non-trivial superbialgebras in dimensions 2, 3 and 4. Moreover we derive a classification of Hopf superalgebras for these dimensions.

The paper is organized as follows. Section 1 reviews definitions and properties of superbialgebras and Hopf superalgebras. Moreover, we provide an outline of the computations leading to classifications. In Section 2, we focus on trivial and connected superbialgebras. We characterize connected superbialgebras and classify connected 2-dimensional superbialgebras and Hopf superalgebras. Sections 3 and 4 deal respectively with classification of 3- and 4-dimensional superbialgebras and Hopf superalgebras. We emphasize on non-trivial superbialgebras, from

which we derive the non-trivial Hopf superalgebras. To this end, we establish classification of 3-dimensional superalgebras and use the 4-dimensional classification due to Armour, Chen and Zhang [7].

Throughout this paper, we work over $\mathbb{K}$, an algebraically closed field of characteristic zero. Unless otherwise specified, we will denote the multiplication by μ or $\cdot$, and the unit element by $\eta(1) = e_1 = e_1^0 = 1$. For simplicity, we will use just product (or multiplication), unit, counit and coproduct (or comultiplication) in the sense of even morphism (i.e. we will drop the suffix 'super'). We also would like to mention that in the last section, most of the results are obtained by using the computer algebra system Mathematica. The program we have used is available.

1 Definitions of superbialgebras and Hopf superalgebras

In this section, we summarize definitions and properties of superbialgebras and Hopf superalgebras. For more details, we refer to [3, 18].

A *superspace* A is a $\mathbb{K}$-vector space endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading, in other words, it writes as a direct sum of two vector spaces $A = A_0 \oplus A_1$ such as A_0 is the even part and A_1 is the odd part. Elements of A_0 (resp. A_1) are called *even homogeneous* (resp. *odd homogeneous*). If $a \in A_0$, we set $|a| = \deg(a) = 0$ and if $a \in A_1$, $|a| = \deg(a) = 1$. Notice that some authors denote the dimension of A by $i|j$, such that $i = n_0$ and $j = n_1$, where $n_0 = \dim A_0$ and $n_1 = \dim A_1$. But in this paper we preserve this notation for the superalgebra obtained by the ith algebra. Notice that the following definitions could be described in a symmetric category of superspaces.

Definition 1.1. A *superalgebra* is a triple (A, μ, η) where A is a superspace, $\mu : A \otimes A \to A$ (multiplication) and $\eta : \mathbb{K} \to A$ (unit) are two superspace morphisms satisfying

$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu) \qquad \text{(associativity)}, \tag{1.1}$$
$$\mu \circ \eta \otimes \mathrm{id}_A = \mu \circ \mathrm{id}_A \otimes \eta \qquad \text{(unity)}. \tag{1.2}$$

A superalgebra A is called *commutative* if the multiplication satisfies $\mu \circ \tau = \mu$ where τ is a superflip. In other words, for all homogeneous elements $a, b \in A$, $\mu(a \otimes b) = (-1)^{|a||b|}\mu(b \otimes a)$.

Let (A, μ_A, η_A), (B, μ_B, η_B) be two superalgebras. A map $f : A \to B$ is a *superalgebra morphism* if it is a superspace morphism satisfying

$$f \circ \mu_A = \mu_B \circ f \otimes f \qquad \text{and} \qquad f \circ \eta_A = \eta_B. \tag{1.3}$$

Definition 1.2. A *supercoalgebra* is a triple (C, Δ, ε) where C is a superspace such that $\Delta : C \to C \otimes C$ (comultiplication or coproduct) and $\varepsilon : C \to \mathbb{K}$ (counit) are two superspace morphisms satisfying

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta \qquad \text{(coassociativity)}, \tag{1.4}$$
$$(\varepsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \varepsilon) \circ \Delta \qquad \text{(counity)}. \tag{1.5}$$

We use the Sweedleer's notation for the coproduct, we set for all $x \in C$

$$\Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)}, \qquad (\mathrm{id} \otimes \Delta) \circ \Delta(x) = (\Delta \otimes \mathrm{id}) \circ \Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} \otimes x^{(3)}.$$

A supercoalgebra C is said to be *cocommutative* if the comultiplication satisfies $\Delta = \tau \circ \Delta$, where τ is the superflip, that is

$$\forall\, x \in C, \qquad \Delta(x) = \sum_{(x)} x^{(1)} \otimes x^{(2)} = \sum_{(x)} (-1)^{|x^{(1)}||x^{(2)}|} x^{(2)} \otimes x^{(1)}.$$

Let $(A, \Delta_A, \varepsilon_A)$, $(B, \Delta_B, \varepsilon_B)$ be two supercoalgebras. The map $f : (A, \Delta_A, \varepsilon_A) \to (B, \Delta_B, \varepsilon_B)$ is a *supercoalgebra morphism* if it is a superspace morphism and satisfies

$$f \otimes f \circ \Delta_A = \Delta_B \circ f \qquad \text{and} \qquad \varepsilon_B \circ f = \varepsilon_A. \tag{1.6}$$

Now, we consider a superbialgebra structure which is obtained by combining superalgebras and supercoalgebras. Moreover, if a superbialgebra has an anti-superbialgebra morphism satisfying some condition, it is called a Hopf superalgebra.

Definition 1.3. A *superbialgebra* is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$, where (A, μ, η) is a superalgebra and (A, Δ, ε) is a supercoalgebra such that one of these two equivalent compatibility conditions hold:

1) $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{K}$ are superalgebra morphisms,
2) $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are supercoalgebra morphisms.

In other words, Δ (resp. ε) satisfies the following compatibility condition

$$\begin{aligned} &\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta), \qquad \Delta \circ \eta = \eta \otimes \eta \\ &(\text{resp.} \quad \varepsilon \circ \mu = \mu_{\mathbb{K}} \circ (\varepsilon \otimes \varepsilon), \quad \varepsilon \circ \eta = \mathrm{id}_{\mathbb{K}}), \end{aligned} \tag{1.7}$$

where $\mu_{\mathbb{K}}$ is the multiplication of $\mathbb{K}$.

The last condition in (1.7) says that the unit $1 = e_1^0 = \eta(1)$ of the superbialgebra is a group-like element $\Delta(1) = 1 \otimes 1$.

Remark 1.4. Note that if $(A = A_0 \oplus A_1, \mu, \eta, \Delta, \varepsilon)$ is a superbialgebra with $\eta(1) = 1$, a unit element, then we have

$$\varepsilon(A_1) = 0 \qquad \text{and} \qquad \varepsilon(1) = 1.$$

Indeed, if $\varepsilon : A \longrightarrow \mathbb{K}$ is a superspace morphism and $\mathbb{K}$ is a superspace whose odd part is 0, then ε sends the odd part of A to odd part of $\mathbb{K}$ which is 0. For the second assertion, if A is a superbialgebra then ε is a superalgebra morphism and it sends the unit element of A to unit element of $\mathbb{K}$.

If A and B are two superbialgebras over $\mathbb{K}$, we shall call a superspace morphism $f : A \to B$ a *superbialgebra morphism* if it is both a superalgebra morphism and a supercoalgebra morphism.

Definition 1.5. A *Hopf superalgebra* is a superbialgebra admitting an antipode, that is a superspace morphism $S : A \to A$ which satisfies the following condition:

$$\mu \circ (S \otimes \mathrm{id}_A) \circ \Delta = \mu \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \varepsilon. \tag{1.8}$$

A Hopf superalgebra is given by a tuple $H = (A, \mu, \eta, \Delta, \varepsilon, S)$.

Let $(A, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf superalgebra and S its antipode. Then we have the following properties:

1) $S \circ \mu = \mu \circ \tau \circ (S \otimes S)$;
2) $S \circ \eta = \eta$;
3) $\varepsilon \circ S = \varepsilon$;
4) $\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$;
5) if A is commutative or cocommutative then $S \circ S = \mathrm{id}$ where $\mathrm{id} : A \longrightarrow A$ is the identity morphism;

6) if the antipode S of the Hopf superalgebra A, with Δ as comultiplication, is bijective then A is also a Hopf superalgebra, with the opposite comultiplication $\Delta' = \tau \circ \Delta$ and the opposite antipode $S' = S^{-1}$;

7) all finite-dimensional Hopf superalgebras possess bijective antipodes.

Remark 1.6. Some superalgebras do not carry a superbialgebra structure. For example $A = M_2(\mathbb{K})$ is a superalgebra with matrix multiplication and such that the even part is $A_0 = \mathbb{K}e_1^0 \oplus \mathbb{K}\, e_2^0$ and the odd part is $A_1 = \mathbb{K}\, e_1^1 \oplus \mathbb{K}\, e_2^1$, where

$$e_1^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_1^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_2^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Indeed, the compatibility condition between the counit ε and the product $\varepsilon(\mu(x \otimes y)) = \varepsilon(x)\varepsilon(y)$ isn't always satisfied, since in one hand we have $\varepsilon(\mu(e_1^1 \otimes e_2^1)) = \varepsilon(e_2^0) = 0$, because $\varepsilon(e_2^1) = \varepsilon(e_1^1) = 0$, and in the other hand $\varepsilon(\mu(e_2^1 \otimes e_1^1)) = \varepsilon(e_1^0 - e_2^0) = 0$. Then $\varepsilon(e_2^0) = 1$, which leads to a contradiction.

In Sections 3 and 4, we aim to classify n-dimensional non-trivial superbialgebras (resp. Hopf superalgebras), for $n = 3$ and $n = 4$. An n-dimensional superbialgebra (resp. a Hopf superalgebra) is identified to its structure constants with respect to a fixed basis. It turns out that the axioms of superbialgebra structure translate to a system of polynomial equations that define the algebraic variety of n-dimensional superbialgebras which is embedded into $\mathbb{K}^{2n_0^3+6n_0n_1^2+n_0-1}$. The classification requires to solve this algebraic system. The calculations are handled using a computer algebra system. We include in the following an outline of the computation.

1. We provide a list of non-trivial n-dimensional superalgebras:
   - For $n = 2$, the list is taken from [14, 17].
   - For $n = 3$, the list is given in Propositions 3.2 and 3.3.
   - For $n = 4$, the list is taken from [6].

2. Fix a superalgebra A and compute the supercoalgebra's structure constants $\{D^k_{(i,s),(j,t)}\}$ and $\{\xi_i^0\}$, where
$$\Delta(e_i^l) = \sum_{s=0}^{1} \sum_{j=1}^{n_s} \sum_{k=1}^{n_t} D^{(j,s)(k)}_{(i,l)} e_j^s \otimes e_k^t$$
with $t = (l + s) \mod [2]$ and
$$\varepsilon(e_i^l) = \begin{cases} \xi_i^0 & \text{if} \quad l = 0 \quad \text{and} \quad i \neq 1, \\ 0 & \text{if} \quad l = 1, \\ 1 & \text{if} \quad l = 0 \quad \text{and} \quad i = 1 \end{cases}$$
with respect to bases $\{e_i^0\}_{i=1,\dots,n_0}$ and $\{e_i^1\}_{i=1,\dots,n_1}$ of A_0 and A_1, respectively. Supercoalgebra structures on A making it a superbialgebra are in one-to-one correspondence with solutions of the system corresponding to (1.4), (1.5), (1.7).

3. Obtained in (2) is a family $\{A_i\}_{i\in I}$ of superalgebras whose underlying superalgebra is A. We sort them into isomorphism classes. Two superbialgebras, given by their structure constants, are isomorphic if there exist matrices $(T^k_{(i,0)}, T^k_{(i,1)})_{i,k}$ defining a superbialgebra morphism with respect to the basis, that is satisfying $\det(T^k_{(i,0)}) \det(T^k_{(i,1)}) \neq 0$ and (1.3)–(1.6).

4. For each representative of isomorphism classes, determine whether it admits an antipode or not. It follows from existence or not of structure constants $\lambda^k_{(i,s)}$, of a morphism S, $S(e_i^s) = \sum_{k=1}^{n_s} \lambda^k_{(i,s)} e_k^s$, satisfying (1.8).

2 Trivial and connected superbialgebras

We describe in this section some properties of connected superbialgebras, which are superbialgebras with 1-dimensional even part and trivial superbialgebras, which are superbialgebras with trivial odd part.

2.1 Trivial superbialgebras

We have the following obvious result.

Proposition 2.1. *Every finite-dimensional bialgebra (resp. Hopf algebra) is a superbialgebra (resp. Hopf superalgebra), whose odd part is reduced to* $\{0\}$. *This superbialgebra (resp. Hopf superalgebra) is called trivial superbialgebra (resp. trivial Hopf superalgebra).*

The algebraic classification of Hopf algebra were investigated by many authors. Hopf algebras of prime dimensions were classified in [39]. We refer to [37, 38] for the classification of finite-dimensional Hopf algebras up to dimension 11, see also [26]. Semisimple Hopf algebras of dimension 12 were given in [16] and complete classification of dimension 12 provided in [32]. See [8] for dimension 14 and [11] for dimension 20. The minimal dimension where the problem is unsolved is 24. For the classification of 2 and 3-dimensional bialgebras, we refer to [14]. They are based on Gabriel's result of 2 and 3-dimensional algebras [17]. Classification of 4-dimensional superbialgebras will be given in a forthcoming paper.

2.2 Connected superbialgebras

In the following we consider connected superbialgebras whose even part are isomorphic to $\mathbb{K}$. The notions of a connected superalgebra, supercoalgebra and Hopf superalgebra are defined in the same way.

Lemma 2.2. *Let* $A = A_0 \oplus A_1$ *be an* $(n+1)$*-dimensional connected superspace. If* $(A, \mu, \eta, \Delta, \varepsilon)$ *is a superbialgebra with* $1 = \eta(1)$, *then we have*

1) $\mu(x \otimes y) = \mu(y \otimes x) = 0$, $\forall\, x, y \in A_1$,
2) $\Delta(x) = 1 \otimes x + x \otimes 1$, $\forall\, x \in A_1$.

Proof. Let $(x_1, x_2, \ldots, x_n)$ be a basis of A_1, and set $\mu(x_i \otimes x_j) = \alpha_{ij} 1$, $\Delta(x_i) = \sum\limits_{k=1}^{n} (\lambda_{k1} x_k \otimes 1 + \lambda_{1k} 1 \otimes x_k)$, with $\alpha_{ij}, \lambda_{1j}, \lambda_{j1} \in \mathbb{K}$, $\forall\, i, j \in \{1, \ldots, n\}$.

- The compatibility of the counit and the multiplication $\varepsilon(\mu(x_i \otimes x_j)) = \varepsilon(x_i)\varepsilon(x_j)$ and the fact that $\varepsilon(x_i) = 0$, $\forall\, i \in \{1, \ldots, n\}$, show that $\alpha_{ij} = 0$, $\forall\, i, j \in \{1, \ldots, n\}$. Then we have $\mu(x_i \otimes x_j) = \mu(x_j \otimes x_i) = 0$, $\forall\, i, j \in \{1, \ldots, n\}$.
- The compatibility of the counit and the following condition:

$$(\varepsilon \otimes \mathrm{id})(\Delta(x_i)) = (\mathrm{id} \otimes \varepsilon)(\Delta(x_i)) = x_i, \quad \forall\, i \in \{1, \ldots, n\},$$

lead to $(\varepsilon \otimes \mathrm{id})(\Delta(x_i)) = \sum\limits_{k=1}^{n} (\lambda_{k1}\varepsilon(x_k) \otimes 1 + \lambda_{1k}\varepsilon(1) \otimes x_k) = \sum\limits_{k=1}^{n} \lambda_{1k} x_k = x_i$ and

$$(\mathrm{id} \otimes \varepsilon)(\Delta(x_i)) = \sum_{k=1}^{n} (\lambda_{k1} x_k \otimes \varepsilon(1) + \lambda_{1k} 1 \otimes \varepsilon(x_k)) = \sum_{k=1}^{n} \lambda_{k1} x_k = x_i.$$

Then $\lambda_{1k} = \lambda_{k1} = 0$, $\forall\, k \in \{1, \ldots, n\} \setminus \{i\}$ and $\lambda_{1i} = \lambda_{i1} = 1$. So

$$\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i, \qquad \forall\, i \in \{1, \ldots, n\}.$$

■

Notice that the first assertion of this lemma is just Proposition 2.11 in [6] for which the proof is different.

Theorem 2.3. *An n-dimensional connected superbialgebra exists only when $n < 3$.*

Proof. Let $A = A_0 \oplus A_1$ be a superalgebra with multiplication μ and unit 1. Assume $A_0 = \mathbb{K}1$, $\dim A_1 = n-1$ and $(x_1, x_2, \ldots, x_{n-1})$ be a basis of A_1.

According to Lemma 2.2, $\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1$. Compatibility condition leads in one hand to $\Delta(\mu(x_i \otimes x_j)) = 0$, and in the other hand to

$$\begin{aligned}\mu \otimes \mu \circ \tau(\Delta(x_i) \otimes \Delta(x_j)) &= \mu \otimes \mu \circ \tau((1 \otimes x_i + x_i \otimes 1) \otimes (1 \otimes x_j + x_j \otimes 1)) \\ &= \mu \otimes \mu \circ \tau(x_i \otimes 1 \otimes x_j \otimes 1 + x_i \otimes 1 \otimes 1 \otimes x_j + 1 \otimes x_i \otimes x_j \otimes 1 + 1 \otimes x_i \otimes 1 \otimes x_j) \\ &= x_i \otimes x_j - x_j \otimes x_i, \qquad \forall i, j \in \{1, \ldots, n-1\}.\end{aligned}$$

Since $x_i \otimes x_j - x_j \otimes x_i \neq 0$, $\forall\, i \neq j$, therefore, the compatibility condition is satisfied only if $n = 2$. ■

A 2-dimensional superbialgebra is either trivial or connected. For trivial 2-dimensional bialgebras, we refer to [14]. In the following, we provide connected 2-dimensional superbialgebras and Hopf superalgebras classification.

Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a 2-dimensional connected superbialgebra. We set $A_0 = \mathbb{K}$, $A_1 = \text{span}\{x\}$ and $\eta(1) = 1$ its unit element.

Proposition 2.4. *Every 2-dimensional connected superbialgebra is isomorphic to 2-dimensional connected superbialgebra $\mathbb{K}[x]/(x^2)$ with $\deg(x) = 1$ and defined by $\Delta(x) = 1 \otimes x + x \otimes 1$.*

Moreover, it carries a Hopf superalgebra structure with an antipode S defined by $S(1) = 1$ and $S(x) = -x$.

Proof. The multiplication, the comultiplication and the counit are defined according to Lemma 2.2 and with this extra product, coproduct and counit $x \cdot x = \alpha 1$, $\Delta(x) = \beta 1 \otimes x + \gamma x \otimes 1$, where $\alpha, \beta, \gamma \in \mathbb{K}$. Solving the system corresponding to conditions (1.1), (1.2), with respect to structure constants α, β, γ, leads to the result. For the second assertion, we assume that the antipode S is defined as, $S(1) = 1$, $S(x) = \lambda x$, with $\lambda \in \mathbb{K}$. Applying the identity (1.8) to x, we obtain only one non trivial 2-dimensional Hopf superalgebra associated to connected 2-dimensional superbialgebra defined above. The antipode is defined as $S(1) = 1$, $S(x) = -x$. ■

Let $f : A \longrightarrow A$ be a superalgebra homomorphism, this means that f is an even linear map, satisfying $f(1) = 1$ and $f \circ \mu = \mu \circ f \otimes f$. Let us set $f(x) = \alpha x$, where $\alpha \in \mathbb{K}\backslash\{0\}$. A direct calculation shows that $f \circ \mu = \mu \circ f \otimes f$ is satisfied for any $\alpha \neq 0$. Therefore, the automorphism group of the 2-dimensional superbialgebra is the infinite group

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \text{ with } \alpha \in \mathbb{K}\backslash\{0\} \right\}.$$

3 Classification of 3-dimensional superbialgebras and Hopf superalgebras

In dimension 3, there are three cases for $n_0 = \dim A_0$. If $n_0 = 1$, we have connected superbialgebras which are covered by Proposition 2.3. If $n_0 = 3$, we have trivial superbialgebras for which we refer to [14] for the classification of 3-dimensional bialgebras. It remains to study the case $n_0 = 2$.

3.1 Superalgebras

Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a 3-dimensional superbialgebra such that $A = A_0 \oplus A_1$ and $\dim A_0 = 2$. Let $\{1, x, y\}$ be a basis of A, such that $\{1, x\}$ generates the even part A_0 and $\{y\}$ generates the odd part A_1. Assume that $\eta(1) = 1$.

We recall here the classification of 2-dimensional algebras.

Proposition 3.1 ([14, 17]). *There are, up to isomorphism, two 2-dimensional algebras* $A_1 = \mathbb{K}[x]/(x^2)$ *and* $A_2 = \mathbb{K}[x]/(x^2 - x)$.

Since the even part of a superalgebra is an algebra, we fix the even part multiplication to be one of the two 2-dimensional algebras, recalled in Proposition 3.1. We denote by $\mu_{i|j}$ the j^{th} multiplication obtained by extending the multiplication μ_i of a 2-dimensional algebra to the multiplication of a 3-dimensional superalgebra. We consider first the algebra A_1.

Proposition 3.2. *Every non-trivial 3-dimensional superalgebra, where the even part is the 2-dimensional algebra* A_1 *is isomorphic to one of the following pairwise nonisomorphic 3-dimensional superalgebras* $A_{1|1} = \mathbb{K}[x, y]/(x^2, y^2, xy)$ *and* $A_{1|2} = \mathbb{K}[x, y]/(x^2, y^2 - x, xy)$, *where* $\deg(x) = 0$ *and* $\deg(y) = 1$.

Proof. We set

$$x \cdot y = \alpha y, \qquad y \cdot x = \beta y, \qquad y \cdot y = \gamma e_1^0 + \sigma x, \qquad \text{with} \quad \alpha, \beta, \gamma, \sigma \in \mathbb{K}.$$

We have $x \cdot (y \cdot y) = \gamma x$, $(x \cdot y) \cdot y = \alpha\gamma e_1^0 + \alpha\sigma x$, $(y \cdot y) \cdot x = \gamma x$ and $y \cdot (y \cdot x) = \beta\gamma e_1^0 + \beta\sigma x$.

According to associativity and by identification, we obtain $\alpha\gamma = 0$, $\alpha\sigma = \gamma$, $\beta\gamma = 0$, $\beta\sigma = \gamma$. Then $\gamma = 0$ and $y \cdot y = \sigma x$.

Moreover, $x \cdot (x \cdot y) = (x \cdot x) \cdot y = 0$ and $(y \cdot x) \cdot x = y \cdot (x \cdot x) = 0$ lead to $\alpha = \beta = 0$. Finally, it remains to fix σ.

If $\sigma = 0$, we obtain the superalgebra defined by $A_{1|1}$. If $\sigma \neq 0$, then we have the superalgebra which is given by the multiplication μ_σ defined as $\mu_\sigma(x \otimes x) = 0$, $\mu_\sigma(y \otimes x) = 0$, $\mu_\sigma(x \otimes y) = 0$, $\mu_\sigma(y \otimes y) = \sigma x, \sigma \neq 0$. It is isomorphic to $A_{1|2}$ $(\sigma = 1)$. ■

For the second algebra A_2, we obtain three possible extensions as a 3-dimensional superalgebra.

Proposition 3.3. *Every non-trivial 3-dimensional superalgebra where the even part is the 2-dimensional algebra* A_2, *is isomorphic to one of the following pairwise nonisomorphic 3-dimensional superalgebras* $A_{2|1} = \mathbb{K}[x, y]/(x^2 - x, y^2 - x, xy - y)$, $A_{2|2} = \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, xy - y, yx)$ *and* $A_{2|3} = \mathbb{K}[x, y]/(x^2 - x,\ y^2, xy - y)$, *where* $\deg(x) = 0$ *and* $\deg(y) = 1$.

Proof. By associativity conditions $(x \cdot x) \cdot y = x \cdot (x \cdot y)$ and $y \cdot (x \cdot x) = (y \cdot x) \cdot x$, it follows that α and β may have only values 0 or 1.

Similarly, conditions $x \cdot (y \cdot y) = (x \cdot y) \cdot y$ and $y \cdot (y \cdot x) = (y \cdot y) \cdot x$ lead to

$$\alpha\gamma = 0, \qquad \alpha\sigma = \gamma + \sigma, \qquad \beta\gamma = 0, \qquad \beta\sigma = \gamma + \sigma. \tag{3.1}$$

We discuss the two cases $\gamma = 0$ and $\gamma \neq 0$.

1. If $\gamma = 0$, then the system (3.1) reduces to $\sigma\,(\alpha - 1) = 0$ and $\sigma\,(\beta - 1) = 0$. So, we deduce two subcases:

a) If $\sigma \neq 0$, then $\alpha = \beta = 1$. We obtain superalgebras which are isomorphic to $A_{2|1}$ $(\sigma = 1)$.

b) If $\sigma = 0$, we study two cases $\alpha = \beta$ and $\alpha \neq \beta$,

i) If $\alpha = \beta$, that means $(\alpha, \beta) \in \{(0,0),(1,1)\}$. Then, we obtain two superalgebras isomorphic to $A_{2|3}$.

ii) If $\alpha \neq \beta$, that means $(\alpha, \beta) \in \{(0,1),(1,0)\}$. Then, we obtain superalgebras which are isomorphic to $A_{2|2}$.

2. If $\gamma \neq 0$. Then, the system (3.1) reduces to $\alpha = \beta = 0$ and $\sigma = -\gamma$. So, we have superalgebras with multiplications μ_γ defined as

$$\mu_\gamma(x \otimes x) = x, \qquad \mu_\gamma(y \otimes x) = 0, \qquad \mu_\gamma(x \otimes y) = 0, \qquad \mu_\gamma(y \otimes y) = \gamma\,(1 - x)\,,$$

which are isomorphic to the superalgebra $A_{2|1}$. ■

3.2 Superbialgebras and Hopf superalgebras

Now, we construct superbialgebra structures associated to the five superalgebras found above.

Proposition 3.4. *There is no* 3*-dimensional superbialgebra, with* $\dim A_0 = 2$*, associated to superalgebras* $A_{1|1}$ *and* $A_{1|2}$*.*

Proof. Assume $\Delta(x) = \alpha 1 \otimes 1 + \beta 1 \otimes x + \gamma x \otimes 1 + \sigma x \otimes x + \delta y \otimes y$ where $\alpha, \beta, \gamma, \sigma, \delta \in \mathbb{K}$. Since we have $x^2 = 0$, the compatibility condition $\varepsilon(\mu(x \otimes x)) = \varepsilon(x)\varepsilon(x)$, implies $\varepsilon(x) = 0$.

In one hand, the condition $(\varepsilon \otimes \mathrm{id})(\Delta(x)) = (\mathrm{id} \otimes \varepsilon)(\Delta(x)) = x$ implies that $\alpha = 0$ and $\beta = \gamma = 1$. In the other hand, the compatibility condition $\Delta \circ \mu(x \otimes x) = (\mu \otimes \mu) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)(x \otimes x) = 0$, leads to $\beta\gamma = 0$. Therefore we have a contradiction. ■

In the sequel, we consider the superalgebra structures defined by $\mu_{2|1}$, $\mu_{2|2}$ and $\mu_{2|3}$. We denote by $A^k_{i|j}$ the superbialgebra $(A, \mu_{i|j}, \eta, \Delta^k_{i|j}, \varepsilon^k_{i|j})$. We have $A^{\mathrm{cop}} = (A, \mu, \eta, \Delta', \varepsilon)$ where $\Delta' = \Delta \circ \tau$ and τ is the superflip. By direct calculation, we obtain the following pairwise non-isomorphic superbialgebras. For all of them we have $\Delta^k_{i|j}(1) = 1 \otimes 1$.

For superalgebra $A_{2|1}$, we have

1) $A^1_{2|1}$ with $\Delta^1_{2|1}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^1_{2|1}(y) = y \otimes 1 + 1 \otimes y - y \otimes x$, $\varepsilon^1_{2|1}(x) = 0$;
2) $(A^1_{2|1})^{\mathrm{cop}}$.

For superalgebra $A_{2|2}$, we have

1) $A^1_{2|2}$ with $\Delta^1_{2|2}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^1_{2|2}(y) = 1 \otimes y + y \otimes 1 - y \otimes x - x \otimes y$, $\varepsilon^1_{2|2}(x) = 0$;
2) $A^2_{2|2}$ with $\Delta^2_{2|2}(x) = 1 \otimes x + x \otimes 1 - x \otimes x + y \otimes y$, $\Delta^2_{2|2}(y) = 1 \otimes y + y \otimes 1 - y \otimes x - x \otimes y$, $\varepsilon^2_{2|2}(x) = 0$;
3) $A^3_{2|2}$ with $\Delta^3_{2|2}(x) = x \otimes x$, $\Delta^3_{2|2}(y) = y \otimes x + x \otimes y$, $\varepsilon^3_{2|2}(x) = 1$;
4) $A^4_{2|2}$ with $\Delta^4_{2|2}(x) = x \otimes x + y \otimes y$, $\Delta^4_{2|2}(y) = y \otimes x + x \otimes y$, $\varepsilon^4_{2|2}(x) = 1$.

For superalgebra $A_{2|3}$, we have

1) $A^1_{2|3}$ with $\Delta^1_{2|3}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^1_{2|3}(y) = 1 \otimes y + y \otimes 1 - x \otimes y - y \otimes x$, $\varepsilon^1_{2|3}(x) = 0$;
2) $A^2_{2|3}$ with $\Delta^2_{2|3}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^2_{2|3}(y) = 1 \otimes y + y \otimes 1 - x \otimes y$, $\varepsilon^2_{2|3}(x) = 0$;
3) $(A^2_{2|3})^{\mathrm{cop}}$;
4) $A^4_{2|3}$ with $\Delta^4_{2|3}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^4_{2|3}(y) = y \otimes 1 + 1 \otimes y$, $\varepsilon^4_{2|3}(x) = 0$;
5) $A^5_{2|3}$ with $\Delta^5_{2|3}(x) = x \otimes x$, $\Delta^5_{2|3}(y) = y \otimes x + x \otimes y$, $\varepsilon^5_{2|3}(x) = 1$.

Theorem 3.5. *Every non-trivial* 3*-dimensional superbialgebra with* $\dim A_0 = 2$ *is isomorphic to one of the following* 3*-dimensional pairwise non-isomorphic superbialgebras, which are defined above,*

$$A^1_{2|1}, \quad \left(A^1_{2|1}\right)^{\mathrm{cop}}, \quad A^k_{2|2} \ (k = 1, \dots, 4), \quad A^1_{2|3}, \quad A^2_{2|3}, \quad \left(A^2_{2|3}\right)^{\mathrm{cop}}, \quad A^4_{2|3}, \quad A^5_{2|3}.$$

Theorem 3.6. *There is no non-trivial* 3*-dimensional Hopf superalgebra.*

Proof. We check that no one of the superbialgebras may carry a structure of Hopf superalgebra. Let S be an antipode of one of the 3-dimensional superbialgebras defined above. Assume that

$$S(1) = 1, \qquad S(x) = \lambda_1 1 + \lambda_2 x, \qquad S(y) = \lambda_3 y,$$

and S satisfies the identity $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \mu \circ (\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon$.

We apply the identity to x and study two cases.

Case 1: $\varepsilon(x) = 0$. For all 3-dimensional superalgebras in this case, we have

$$\Delta(x) = 1 \otimes x + x \otimes 1 - x \otimes x + \alpha y \otimes y, \qquad \text{such that} \quad \alpha = 0 \quad \text{or} \quad 1.$$

In one hand, we have,

$$\begin{aligned} \mu \circ (S \otimes \mathrm{id}_A) \circ \Delta(x) &= \mu \circ (1 \otimes x + x \otimes 1 - S(x) \otimes x + \alpha S(y) \otimes y) \\ &= x + (\lambda_1 x + \lambda_2 x) - \lambda_1 x - \lambda_2 x = \lambda_1 1 + (1 - \lambda_1) x. \end{aligned}$$

In the other hand, we have $\eta \circ \varepsilon(x) = 0$. Therefore, we have a contradiction.

Case 2: $\varepsilon(x) = 1$. In this case, $\Delta(x)$ must have the form

$$\Delta(x) = x \otimes x + \alpha y \otimes y, \qquad \text{such that} \quad \alpha = 0 \quad \text{or} \quad 1.$$

Then, on one hand, we have,

$$\mu \circ (S \otimes \mathrm{id}_A) \circ \Delta(x) = \mu \circ (S(x) \otimes x + \alpha S(y) \otimes y) = (\lambda_1 + \lambda_2) x.$$

On the other hand, we have $\eta \circ \varepsilon(x) = 1$. So we have a contradiction. ■

4 Classification of 4-dimensional superbialgebras and Hopf superalgebras

Let $A = A_0 \oplus A_1$ be a 4-dimensional superbialgebra. Then one has to consider four cases for $n_0 = \dim A_0$. If $n_0 = 1$, the superbialgebras are connected, then it is covered by Proposition 2.3. If $n_0 = 4$, then they correspond to trivial superbialgebras. In the sequel, we discuss cases $n_0 = 2$ and $n_0 = 3$.

4.1 4-dimensional algebras

In the following, we recall the classification of 4-dimensional algebras given by Gabriel in [17] and the classification of 4-dimensional superalgebras with $n_0 = 2$ and $n_0 = 3$ provided by Armour, Chen and Zhang in [7].

Theorem 4.1. *The following algebras are pairwise non-isomorphic and every* 4*-dimensional algebras is isomorphic to one of these* 19 *algebras:*

1) $\mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$,
2) $\mathbb{K} \times \mathbb{K} \times \mathbb{K}[x]/x^2$,

3) $\mathbb{K}[x]/x^2 \times \mathbb{K}[y]/y^2$,
4) $\mathbb{K} \times \mathbb{K}[x]/x^3$,
5) $\mathbb{K}[x]/x^4$,
6) $\mathbb{K} \times \mathbb{K}[x,y]/(x,y)^2$,
7) $\mathbb{K}[x,y]/(x^2,y^2)$,
8) $\mathbb{K}[x,y]/(x^3,xy,y^2)$,
9) $\mathbb{K}[x,y,z]/(x,y,z)^2$,
10) $M_2 = \begin{pmatrix} \mathbb{K} & \mathbb{K} \\ \mathbb{K} & \mathbb{K} \end{pmatrix}$,
11) $\left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & d \\ c & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} / a,b,c,d \in \mathbb{K} \right\}$,
12) $\Lambda\mathbb{K}^2 =$ *exterior algebra of* $\mathbb{K}^2$,
13) $\mathbb{K} \times \begin{pmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{pmatrix}$,
14) $\left\{ \begin{pmatrix} a & 0 & 0 \\ c & a & 0 \\ d & 0 & b \end{pmatrix} / a,b,c,d \in \mathbb{K} \right\}$,
15) $\left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} / a,b,c,d \in \mathbb{K} \right\}$,
16) $\mathbb{K}[x,y]/(x^2,y^2,yx)$,
17) $\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & d & b \end{pmatrix} / a,b,c,d \in \mathbb{K} \right\}$,
18) $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx-\lambda xy)$, *for* $\lambda \neq -1,0,1$,
19) $\mathbb{K}\langle x,y\rangle/(y^2,x^2+yx,xy+yx)$,

where $\mathbb{K}\langle x,y\rangle$ *is the free associative algebra generated by* x *and* y.

4.2 Superalgebras with $\dim(A_0) = 3$

Let $\{e_1^0, e_2^0, e_3^0, e_1^1\}$ be a basis of the superalgebra A, $A = A_0 \oplus A_1$, such that $A_0 = \text{span}\{e_1^0, e_2^0, e_3^0\}$, $A_1 = \text{span}\{e_1^1\}$ and e_1^0 be the unit element of the superalgebra. We recall the multiplication of all possible graduation obtained from a fixed algebra, we denote by $\mu_{i|j}$ the multiplication of each 4-dimensional superalgebra in the case $\dim A_0 = 3$. We preserve the notation used in [7], a superalgebra $i|j$ means the j^{th} superalgebra obtained by a fixed i^{th} 4-dimensional algebra.

Proposition 4.2 ([7]). *Let* $\mathbb{K}$ *be an algebraically closed field of characteristic* 0*, suppose that* A *is a* 4*-dimensional superalgebra with* $\dim A_0 = 3$*. Then* A *is isomorphic to one of the superalgebra in the following pairwise nonisomorphic families:*

1|1: $\mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$, $e_1^0 = (1,1,1,1)$, $e_2^0 = (1,0,0,0)$, $e_3^0 = (0,0,1,1)$, $e_1^1 = (0,0,1,-1)$;

2|1: $\mathbb{K} \times \mathbb{K} \times \mathbb{K}[x]/x^2$, $e_1^0 = (1,1,1)$, $e_2^0 = (1,0,0)$, $e_3^0 = (0,1,0)$, $e_1^1 = (0,0,x)$;

2|2: $\mathbb{K} \times \mathbb{K} \times \mathbb{K}[x]/x^2$, $e_1^0 = (1,1,1)$, $e_2^0 = (1,1,0)$, $e_3^0 = (0,0,x)$, $e_1^1 = (1,-1,0)$;

3|1: $\mathbb{K}[x]/x^2 \times \mathbb{K}[y]/y^2$, $e_1^0 = (1,1)$, $e_2^0 = (1,0)$, $e_3^0 = (x,0)$, $e_1^1 = (0,y)$;

4|1: $\mathbb{K} \times \mathbb{K}[x]/x^3$, $e_1^0 = (1,1)$, $e_2^0 = (1,0)$, $e_3^0 = (0,x^2)$, $e_1^1 = (0,x)$;

6|1: $\mathbb{K} \times \mathbb{K}[x,y]/(x,y)^2 : e_1^0 = (1,1)$, $e_2^0 = (1,0)$, $e_3^0 = (0,x)$, $e_1^1 = (0,y)$;

7|1: $\mathbb{K}[x,y]/(x^2,y^2)$, $e_1^0 = 1$, $e_2^0 = x+y$, $e_3^0 = xy$, $e_1^1 = x-y$;

8|1: $\mathbb{K}[x,y]/(x^3,xy,y^2)$, $e_1^0=1$, $e_2^0=x$, $e_3^0=x^2$, $e_1^1=y$;

8|2: $\mathbb{K}[x,y]/(x^3,xy,y^2)$, $e_1^0=1$, $e_2^0=x^2$, $e_3^0=y$, $e_1^1=x$;

9|1: $\mathbb{K}[x,y,z]/(x,y,z)^2$, $e_1^0=1$, $e_2^0=x$, $e_3^0=y$, $e_1^1=z$;

11|1: $\left\{\begin{pmatrix} a & 0 & 0 & 0\\ 0 & a & 0 & d\\ c & 0 & b & 0\\ 0 & 0 & 0 & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$;

13|1: $\mathbb{K}\times\begin{pmatrix}\mathbb{K} & \mathbb{K}\\ 0 & \mathbb{K}\end{pmatrix}=\left\{\left(a,\begin{pmatrix} b & c\\ 0 & d\end{pmatrix}\right)/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\left(1,\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right)$,
$e_2^0=\left(0,\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\right)$, $e_3^0=\left(0,\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\right)$, $e_1^1=\left(0,\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\right)$;

14|1: $\left\{\begin{pmatrix} a & 0 & 0\\ c & a & 0\\ d & 0 & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$;

14|2: $\left\{\begin{pmatrix} a & 0 & 0\\ c & a & 0\\ d & 0 & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$;

15|1: $\left\{\begin{pmatrix} a & c & d\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$;

15|2: $\left\{\begin{pmatrix} a & c & d\\ 0 & a & 0\\ 0 & 0 & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$;

17|1: $\left\{\begin{pmatrix} a & 0 & 0\\ 0 & a & 0\\ c & d & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$,
$e_3^0=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$.

4.3 Superbialgebras and Hopf superalgebras with $\dim(A_0) = 3$

We look for all possible superbialgebras which could be obtained from a fixed superalgebra. The computation are done using a computer algebra system. For the convenience of the presentation, we only summarize the results in the following proposition. The notations and details are collected in Appendix A.1.

Proposition 4.3. *Let $(A, \mu, \eta, \Delta, \varepsilon)$ be a 4-dimensional superbialgebra with $\dim A_0 = 3$. Then A is isomorphic to one of the superbialgebra in the following pairwise nonisomorphic families:*

superalgebra	*associated superbialgebras*
1|1	$\left(A, \mu_{1\|1}, \eta, \Delta^k_{1\|1}, \varepsilon^k_{1\|1}\right)$, $k = 1, \dots, 12$
2|1	$\left(A, \mu_{2\|1}, \eta, \Delta^k_{2\|1}, \varepsilon^k_{2\|1}\right)$, $k = 1, \dots, 22$
4|1	$\left(A, \mu_{4\|1}, \eta, \Delta^k_{4\|1}, \varepsilon^k_{4\|1}\right)$, $k = 1, 2, 3$
6|1	$\left(A, \mu_{6\|1}, \eta, \Delta^k_{6\|1}, \varepsilon^k_{6\|1}\right)$, $k = 1, \dots, 18$
13|1	$\left(A, \mu_{13\|1}, \eta, \Delta^k_{13\|1}, \varepsilon^k_{13\|1}\right)$, $k = 1, \dots, 21$
14|1	$\left(A, \mu_{14\|1}, \eta, \Delta^k_{14\|1}, \varepsilon^k_{14\|1}\right)$, $k = 1, \dots, 9$
14|2	$\left(A, \mu_{14\|2}, \eta, \Delta^k_{14\|2}, \varepsilon^k_{14\|2}\right)$, $k = 1, \dots, 4$
15|1	$\left(A, \mu_{15\|1}, \eta, \Delta^k_{15\|1}, \varepsilon^k_{15\|1}\right)$, $k = 1, \dots, 9$
15|2	$\left(A, \mu_{15\|2}, \eta, \Delta^k_{15\|2}, \varepsilon^k_{15\|2}\right)$, $k = 1, \dots, 4$
17|1	$\left(A, \mu_{17\|1}, \eta, \Delta^k_{17\|1}, \varepsilon^k_{17\|1}\right)$, $k = 1, \dots, 11$

Remark 4.4. There is no 4-dimensional superbialgebra with $\dim A_0 = 3$ and underlying multiplications $\mu_{2|2}$, $\mu_{3|1}$, $\mu_{7|1}$, $\mu_{8|1}$, $\mu_{8|2}$, $\mu_{9|1}$, $\mu_{11|1}$.

Now, we look for Hopf superalgebra structures. For a fixed superbialgebra defined above, we add the antipode's property. It turns out that there exists only one non-trivial 4-dimensional Hopf superalgebra in the case $\dim A_0 = 3$. It corresponds to the algebra (1|1) and comultiplication $\Delta^2_{1|1}$, where $A = \mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$. We set $x = e^0_1 - 2e^0_2 - e^0_3 = (-1, 1, 0, 0)$ and $y = e^1_1 = (0, 0, 1, -1)$ with $\deg(x) = 0$, $\deg(y) = 1$. It leads to basis $\{1, x, x^2, y\}$ and the algebra may be written as $\mathbb{K}[x, y]/(x^2 + y^2 - 1, xy)$ with $\deg(x) = 0$ and $\deg(y) = 1$.

Proposition 4.5. *Every non-trivial 4-dimensional Hopf superalgebra where $\dim A_0 = 3$ is isomorphic to the 4-dimensional Hopf superalgebra $\mathbb{K}[x, y]/(x^2 + y^2 - 1, xy)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that*

$$\begin{aligned} &\Delta(x) = x \otimes x - \alpha y \otimes y, \qquad \varepsilon(x) = 1, \qquad S(x) = x, \\ &\Delta(y) = x \otimes y + y \otimes x, \qquad \varepsilon(y) = 0, \qquad S(y) = \alpha y. \end{aligned}$$

where α is a primitive 4^{th} root of unity.

4.4 Superalgebras with $\dim(A_0) = 2$

Let $\{e^0_1, e^0_2, e^1_1, e^1_2\}$ be a basis of the underlying superspace A, such that $\{e^0_1, e^0_2\}$ is a basis of the even part, $\{e^1_1, e^1_2\}$ a basis of the odd part and e^0_1 is the unit of the superalgebra.

Proposition 4.6 ([7])**.** *Let $\mathbb{K}$ be an algebraically closed field, suppose that A is a 4-dimensional superalgebra with $\dim A_0 = 2$. Then A is isomorphic to one of the superalgebra in the following pairwise non-isomorphic families:*

1|2: $\mathbb{K} \times \mathbb{K} \times \mathbb{K} \times \mathbb{K}$, $e^0_1 = (1, 1, 1, 1)$, $e^0_2 = (1, 1, 0, 0)$, $e^1_1 = (1, -1, 0, 0)$, $e^1_2 = (0, 0, 1, -1)$;

2|3: $\mathbb{K}\times\mathbb{K}\times\mathbb{K}[x]/x^2$, $e_1^0=(1,1,1)$, $e_2^0=(1,1,0)$, $e_1^1=(1,-1,0)$, $e_2^1=(0,0,x)$;

3|2: $\mathbb{K}[x]/x^2\times\mathbb{K}[y]/y^2$, $e_1^0=(1,1)$, $e_2^0=(1,0)$, $e_1^1=(x,0)$, $e_2^1=(0,y)$;

3|3: $\mathbb{K}[x]/x^2\times\mathbb{K}[y]/y^2$, $e_1^0=(1,1)$, $e_2^0=(x,y)$, $e_1^1=(1,-1)$, $e_2^1=(x,-y)$;

5|1: $\mathbb{K}[x]/x^4$, $e_1^0=1$, $e_2^0=x^2$, $e_1^1=x$, $e_2^1=x^3$;

6|2: $\mathbb{K}\times\mathbb{K}[x,y]/(x,y)^2$, $e_1^0=(1,1)$, $e_2^0=(1,0)$, $e_1^1=(0,x)$, $e_2^1=(0,y)$;

7|2: $\mathbb{K}[x,y]/(x^2,y^2)$, $e_1^0=1$, $e_2^0=x$, $e_1^1=y$, $e_2^1=xy$;

7|3: $\mathbb{K}[x,y]/(x^2,y^2)$, $e_1^0=1$, $e_2^0=xy$, $e_1^1=x$, $e_2^1=y$;

8|3: $\mathbb{K}[x,y]/(x^3,xy,y^2)$, $e_1^0=1$, $e_2^0=x^2$, $e_1^1=x$, $e_2^1=y$;

9|2: $\mathbb{K}[x,y,z]/(x,y,z)^2$, $e_1^0=1$, $e_2^0=x$, $e_1^1=y$, $e_2^1=z$;

10|1: $M_2=\begin{pmatrix}\mathbb{K}&\mathbb{K}\\ \mathbb{K}&\mathbb{K}\end{pmatrix}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $e_2^0=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $e_1^1=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_2^1=\begin{pmatrix}0&0\\1&0\end{pmatrix}$;

11|2: $\left\{\begin{pmatrix}a&0&0&0\\0&a&0&d\\c&0&b&0\\0&0&0&b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $e_2^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$, $e_1^1=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$, $e_2^1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$;

11|3: $\left\{\begin{pmatrix}a&0&0&0\\0&a&0&d\\c&0&b&0\\0&0&0&b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$, $e_2^0=\begin{pmatrix}0&0&0&0\\0&0&0&1\\1&0&0&0\\0&0&0&0\end{pmatrix}$, $e_1^1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$, $e_2^1=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\1&0&0&0\\0&0&0&0\end{pmatrix}$;

12|1: $\Lambda\mathbb{K}^2=\mathbb{K}\langle x,y\rangle/(x^2,y^2,xy+yx)$, $e_1^0=1$, $e_2^0=x$, $e_1^1=y$, $e_2^1=xy$;

12|2: $\Lambda\mathbb{K}^2=\mathbb{K}\langle x,y\rangle/(x^2,y^2,xy+yx)$, $e_1^0=1$, $e_2^0=xy$, $e_1^1=x$, $e_2^1=y$;

14|3: $\left\{\begin{pmatrix}a&0&0\\c&a&0\\d&0&b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $e_2^0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$, $e_1^1=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}$, $e_2^1=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$;

15|3: $\left\{\begin{pmatrix}a&c&d\\0&a&0\\0&0&b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $e_2^0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$, $e_1^1=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$, $e_2^1=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$;

16|1: $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx)$, $e_1^0=1$, $e_2^0=x$, $e_1^1=y$, $e_2^1=xy$;

16|2: $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx)$, $e_1^0=1$, $e_2^0=y$, $e_1^1=x$, $e_2^1=xy$;

16|3: $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx)$, $e_1^0=1$, $e_2^0=xy$, $e_1^1=x$, $e_2^1=y$;

17|2: $\left\{\begin{pmatrix} a & 0 & 0\\ 0 & a & 0\\ c & d & b\end{pmatrix}/a,b,c,d\in\mathbb{K}\right\}$, $e_1^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $e_2^0=\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$, $e_1^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$, $e_2^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$;

(18; λ|1): $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx-\lambda xy)$, *where* $\lambda\in\mathbb{K}$ *with* $\lambda\neq -1,0,1$, $e_1^0=1$, $e_2^0=x$, $e_1^1=y$, $e_2^1=xy$;

(18; λ|2): $\mathbb{K}\langle x,y\rangle/(x^2,y^2,yx-\lambda xy)$, *where* $\lambda\in\mathbb{K}$ *with* $\lambda\neq -1,0,1$, $e_1^0=1$, $e_2^0=xy$, $e_1^1=x$, $e_2^1=y$;

19|1: $\mathbb{K}\langle x,y\rangle/(y^2,x^2+yx,xy+yx)$, $e_1^0=1$, $e_2^0=xy$, $e_1^1=x$, $e_2^1=y$.

4.5 Superbialgebras and Hopf superalgebras with $\dim(A_0)=2$

Now, we compute all superbialgebra structures associated to superalgebras described above where $\dim A_0=2$. We give fairly basic description of the results, all details are given in Appendix A.2. We denote by $\mu_{i|j}$ the multiplication in $A_{i|j}$ with respect to the basis described above.

Proposition 4.7. *There is no* 4*-dimensional superbialgebra with* $\dim A_0=2$ *endowed with one of these multiplications* $\mu_{1|2}$, $\mu_{3|3}$, $\mu_{5|1}$, $\mu_{7|2}$, $\mu_{7|3}$, $\mu_{8|3}$, $\mu_{10|1}$, $\mu_{11|3}$, $\mu_{12|1}$, $\mu_{16|1}$, $\mu_{16|2}$, $\mu_{16|3}$, $\mu_{18|1}$, $\mu_{18|2}$, $\mu_{19|1}$.

Proposition 4.8. *Let* $(A,\mu,\eta,\Delta,\varepsilon)$ *be a* 4*-dimensional superbialgebra with* $\dim A_0=2$*, then* A *is isomorphic to one of the superbialgebra in the following pairwise nonisomorphic families:*

superalgebra	*associated superbialgebras*
2|3	$\left(A,\mu_{2\|3},\eta,\Delta^k_{2\|3},\varepsilon^k_{2\|3}\right)$, $k=1,\dots,4$
3|2	$\left(A,\mu_{3\|2},\eta,\Delta^k_{3\|2},\varepsilon^k_{3\|2}\right)$, $k=1,\dots,9$
6|2	$\left(A,\mu_{6\|2},\eta,\Delta^k_{6\|2},\varepsilon^k_{6\|2}\right)$, $k=1,\dots,11$
11|2	$\left(A,\mu_{11\|2},\eta,\Delta^1_{11\|2},\varepsilon^1_{11\|2}\right)$
12|2	$\left(A,\mu_{12\|2},\eta,\Delta^1_{12\|2},\varepsilon^1_{12\|2}\right)$
14|3	$\left(A,\mu_{14\|3},\eta,\Delta^1_{14\|3},\varepsilon^1_{14\|3}\right)$, $k=1,\dots,7$
15|3	$\left(A,\mu_{15\|3},\eta,\Delta^1_{15\|3},\varepsilon^1_{15\|3}\right)$, $k=1,\dots,7$
17|2	$\left(A,\mu_{17\|2},\eta,\Delta^1_{17\|2},\varepsilon^1_{17\|2}\right)$, $k=1,2$

It turns out that the only superbialgebras, with $\dim A_0=2$, which can have a non-trivial Hopf superalgebra structure are (3|2), (11|2) and (12|2).

Proposition 4.9. *Every non-trivial* 4*-dimensional Hopf superalgebra where* $\dim A_0=2$ *is isomorphic to one of the following pairwise non-isomorphic Hopf superalgebras:*

1) $\mathcal{H}_1=\mathbb{K}\langle x,y\rangle/(x^2-x,y^2,yx)$ *with* $\deg(x)=0$, $\deg(y)=1$ *and such that*

$$\Delta(x)=1\otimes x+x\otimes 1-2x\otimes x,\qquad \Delta(y)=1\otimes y+y\otimes 1,$$
$$\varepsilon(x)=\varepsilon(y)=0,\qquad S(x)=x,\qquad S(y)=-y;$$

2) $\mathcal{H}_2 = \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, xy - yx - y)$ *with* $\deg(x) = 0$, $\deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \qquad \Delta(y) = 1 \otimes y + y \otimes 1 - 2x \otimes xy - 2xy \otimes x,$$
$$\varepsilon(x) = \varepsilon(y) = 0, \qquad S(x) = x, \qquad S(y) = y;$$

3) $\mathcal{H}_3 = \mathbb{K}\langle x, y\rangle/(x^2, y^2, xy + yx)$ *with* $\deg(x) = \deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1, \qquad \Delta(y) = 1 \otimes y + y \otimes 1,$$
$$\varepsilon(x) = \varepsilon(y) = 0, \qquad S(x) = -x, \qquad S(y) = -y;$$

4) $\mathcal{H}_4 = \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, yx)$ *with* $\deg(x) = 0$, $\deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \qquad \Delta(y) = 1 \otimes y + y \otimes 1 - 2x \otimes y,$$
$$\varepsilon(x) = \varepsilon(y) = 0, \qquad S(x) = x, \qquad S(y) = 2xy - y,$$

where $\mathbb{K}\langle x, y\rangle$ *stands for noncommutative polynomials.*

Proof. The Hopf superalgebra $\mathcal{H}_1$ corresponds to $A^2_{3|2}$ (see Appendix A.2) with $x = e^0_2$, $y = e^1_1 + e^1_2$.

The Hopf superalgebra $\mathcal{H}_2$ corresponds to $A^1_{11|2}$ with $x = e^0_2$, $y = e^1_1 - e^1_2$.

The Hopf superalgebra $\mathcal{H}_3$ corresponds to $A^1_{12|2}$ with $x = e^1_1$, $y = e^1_2$.

The Hopf superalgebra $\mathcal{H}_4$ corresponds to $A^1_{3|2}$ with $x = e^0_2$, $y = e^1_1 + e^1_2$. ■

Therefore, gathering the results corresponding to the case $dim A_0 = 3$ and $dim A_0 = 2$ we obtain five 4-dimensional Hopf superalgebras.

Theorem 4.10. *Every non-trivial* 4*-dimensional Hopf superalgebra is isomorphic to one of the following Hopf superalgebras*

$$\mathcal{H}_1 \cong \mathbb{K}[\mathbb{Z}/2\mathbb{Z}] \otimes \Lambda\mathbb{K}, \qquad \mathcal{H}_2 \cong \mathbb{K}[\mathbb{Z}/2\mathbb{Z}] \rtimes_\sigma \Lambda\mathbb{K},$$
$$\mathcal{H}_3 \cong \Lambda\mathbb{K}^2, \qquad \mathcal{H}_4 = \mathcal{H}_2^* \qquad \textit{and} \qquad \mathcal{H}_5,$$

where $\sigma : \mathbb{Z}/2\mathbb{Z} \to GL(\mathbb{K})$ *is the non-trivial action of* $\mathbb{Z}/2\mathbb{Z}$ *on* $\mathbb{K}$, $\mathcal{H}_2^*$ *is the dual of* $\mathcal{H}_2$ *and* $\mathcal{H}_5$ *is defined as* $\mathbb{K}[x, y]/(x^2 + y^2 - 1, xy)$ $(\deg(x) = 0, \deg(y) = 1)$ *such that*

$$\Delta(x) = x \otimes x - \alpha y \otimes y, \qquad \varepsilon(x) = 1, \qquad S(x) = x,$$
$$\Delta(y) = x \otimes y + y \otimes x, \qquad \varepsilon(y) = 0, \qquad S(y) = \alpha y.$$

with $\alpha^4 = 1$.

Proof. The first three Hopf superalgebras are cocommutative. We obtain the isomorphisms, thanks to Kostant's theorem. However, since $\mathcal{H}_4$ is commutative, its dual is cocommutative, it turns out that we have $\mathcal{H}_4 = \mathcal{H}_2^*$. The Hopf superalgebra $\mathcal{H}_5$ corresponds to $A_{1|1}$ (Proposition 4.5), it is not cocommutative and is the only one whose even part is 3-dimensional. ■

In the following table, we collect the results obtained for 4-dimensional superbialgebras and Hopf superalgebras. Notice that algebras are those classified by Gabriel, see Theorem 4.1. The superalgebras $(i|j)$ denote j^{th} graduation of the i^{th} algebra, obtained by Armour, Chen and Zhang, see Proposition 4.2 for the first case ($\dim A_0 = 3$) and Proposition 4.6 for the second case ($\dim A_0 = 2$). We quote below the number of corresponding superbialgebras and Hopf superalgebras.

algebra	superalgebra	♯ superbialgebras	♯ Hopf superalgebras
1	1|1	12	1
	1|2	0	0
2	2|1	22	0
	2|2	0	0
	2|3	4	0
3	3|1	0	0
	3|2	9	2
	3|3	0	0
4	4|1	3	0
5	5|1	0	0
6	6|1	18	0
	6|2	11	0
7	7|1	0	0
	7|2	0	0
	7|3	0	0
8	8|1	0	0
	8|2	0	0
	8|3	0	0
9	9|1	0	0
	9|2	0	0
10	10|1	0	0
11	11|1	0	0
	11|2	1	1
	11|3	0	0
12	12|1	0	0
	12|2	1	1
13	13|1	21	0
14	14|1	9	0
	14|2	4	0
	14|3	7	0
15	15|1	9	0
	15|2	4	0
	15|3	7	0
16	16|2	0	0
	16|1	0	0
	16|3	0	0
17	17|1	11	0
	17|2	2	0
$(18;\lambda)$	$(18;\lambda)\|1$	0	0
	$(18;\lambda)\|2$	0	0
19	19|1	0	0

A Appendix

We list in the following supercoalgebras associated to a given 4-dimensional superalgebra A, such that $A = A_0 \oplus A_1$. We denote the comultiplication by $\Delta^k_{i|j}$ and the counit by $\varepsilon^k_{i|j}$, where i indicates the item of the 4-dimensional algebra listed in Theorem 4.1, and $A_{i|j}$ denotes the superalgebra obtained from the i^{th} algebra, see Proposition 4.2 for the first case ($\dim A_0 = 3$)

and Proposition 4.6 for the second case ($\dim A_0 = 2$). The exponent k indicates the item of the comultiplication and counit which combined with the multiplication of superalgebra $A_{i|j}$ and the unit η provide a 4-dimensional superbialgebra. Recall that for all of them we have $\eta(1) = 1$ (unit element), $\Delta(1) = 1 \otimes 1$, $\varepsilon(1) = 1$ and $\varepsilon(A_1) = 0$. In the sequel, we denote by $A^k_{i|j}$ the superbialgebra $(A, \mu_{i|j}, \eta, \Delta^k_{i|j}, \varepsilon^k_{i|j})$. In order to simplify the superbialgebra in this appendix, we change the variables for superalgebras $A_{1|1}$, $A_{2|1}$, $A_{3|2}$, $A_{6|1}$, $A_{13|1}$, $A_{11|2}$, $A_{12|2}$. For the superalgebras $A_{1|1}$, $A_{3|2}$, $A_{11|2}$, $A_{12|2}$ we change the variables as mentioned before Proposition 4.5 and in the proof of Proposition 4.9. For $A_{2|1}$ and $A_{13|1}$ we use $x = e^0_2 - e^0_3$, $y = e^1_1$, and for $A_{6|1}$ we use $x = e^0_2 + e^0_3$, $y = e^1_1$. According to the multiplication of each superalgebra we obtain that $A_{2|1} \cong \mathbb{K}[x,y]/(x^3 - x, xy, y^2)$, $A_{6|1} \cong \mathbb{K}[x,y]/(x^3 - x^2, xy, y^2)$, $A_{13|1} \cong \mathbb{K}\langle x,y\rangle/(x^3 - x, xy + yx, xy - y, y^2)$. For the remaining superalgebras we change just the notation of basis vectors. For $\dim(A_0) = 3$, we consider the basis $\{1 = e^0_1, x = e^0_2, y = e^0_3, z = e^1_1\}$ and for $\dim(A_0) = 2$ the basis $\{1 = e^0_1, x = e^0_2, y = e^1_1, z = e^1_2\}$.

A.1 Case $\dim A_0 = 3$

Superalgebra $A_{1|1} \cong \mathbb{K}[x,y]/(x^2 + y^2 - 1, xy)$ with $\deg(x) = 0$ and $\deg(y) = 1$, we have

1) $A^1_{1|1}$ with $\Delta^1_{1|1}(x) = \frac{1}{2}(x \otimes 1 + x^2 \otimes 1 + x^2 \otimes x - x \otimes x)$,
$\Delta^1_{1|1}(y) = y \otimes 1 + \frac{1}{2}x^2 \otimes y - \frac{1}{2}x \otimes y$, $\varepsilon^1_{1|1}(x) = -1$;
2) $A^2_{1|1}$ with $\Delta^2_{1|1}(x) = x \otimes x - \alpha y \otimes y$, $\Delta^2_{1|1}(y) = x \otimes y + y \otimes x$, $\varepsilon^2_{1|1}(x) = 1$,
where α is a primitive 4th root of unity of $\mathbb{K}$;
3) $A^3_{1|1}$ with $\Delta^3_{1|1}(x) = x \otimes x$, $\Delta^3_{1|1}(y) = 1 \otimes y + y \otimes x$, $\varepsilon^3_{1|1}(x) = 1$;
4) $A^4_{1|1}$ with $\Delta^4_{1|1}(x) = \frac{1}{2}(x^2 \otimes x^2 + x \otimes x^2 + x^2 \otimes x - x \otimes x)$,
$\Delta^4_{1|1}(y) = 1 \otimes y + y \otimes x^2$, $\varepsilon^4_{1|1}(x) = -1$;
5) $A^5_{1|1}$ with $\Delta^5_{1|1}(x) = \frac{1}{2}(x \otimes 1 + 1 \otimes x - 1 \otimes x^2 - x^2 \otimes 1 + x^2 \otimes x^2 + x \otimes x)$,
$\Delta^5_{1|1}(y) = y \otimes 1 + \frac{1}{2}(y \otimes x + x \otimes y - y \otimes x^2 + x^2 \otimes y)$, $\varepsilon^5_{1|1}(x) = 1$;
6) $A^6_{1|1}$ with $\Delta^6_{1|1}(x) = \frac{1}{2}(x \otimes x + x \otimes x^2 + x^2 \otimes x - x^2 \otimes x^2)$,
$\Delta^6_{1|1}(y) = 1 \otimes y + y \otimes x^2$, $\varepsilon^6_{1|1}(x) = 1$;
7) $A^7_{1|1}$ with $\Delta^7_{1|1}(x) = -1 \otimes 1 + \frac{1}{2}(x \otimes x + x \otimes 1 + 1 \otimes x + 1 \otimes x^2 + x^2 \otimes 1 - x^2 \otimes x^2)$,
$\Delta^7_{1|1}(y) = \frac{1}{2}(x \otimes y + y \otimes x + x^2 \otimes y + y \otimes x^2)$, $\varepsilon^7_{1|1}(x) = 1$;
8) $A^8_{1|1}$ with $\Delta^8_{1|1}(x) = x \otimes x$, $\Delta^8_{1|1}(y) = y \otimes 1 + x^2 \otimes y$, $\varepsilon^8_{1|1}(x) = 1$;
9) $A^9_{1|1} = (A^4_{1|1})^{\mathrm{cop}}$, $A^{10}_{1|1} = (A^1_{1|1})^{\mathrm{cop}}$, $A^{11}_{1|1} = (A^3_{1|1})^{\mathrm{cop}}$, $A^{12}_{1|1} = (A^5_{1|1})^{\mathrm{cop}}$.

Superalgebra $A_{2|1} \cong \mathbb{K}[x,y]/(x^3 - x, y^2, xy)$ with $\deg(x) = 0$ and $\deg(y) = 1$, $\varepsilon^k_{2|1}(x) = 1$ for $k = 1, \ldots, 22$:

1) $A^k_{2|1}$ with $\Delta^k_{2|1}(x) = x \otimes x$ for $k = 1, \ldots, 5$:
   - $\Delta^1_{2|1}(y) = y \otimes x + x \otimes y$,
   - $\Delta^2_{2|1}(y) = x \otimes y + y \otimes x^2$,
   - $\Delta^3_{2|1}(y) = y \otimes x^2 + x^2 \otimes y$,
   - $\Delta^4_{2|1}(y) = 1 \otimes y + y \otimes x$,
   - $\Delta^5_{2|1}(y) = 1 \otimes y + y \otimes x^2$;
2) $A^6_{2|1}$ with $\Delta^6_{2|1}(x) = \frac{1}{2}(x \otimes 1 + x \otimes x + x^2 \otimes x - x^2 \otimes 1)$, $\Delta^6_{2|1}(y) = \frac{1}{2}(y \otimes x + x \otimes y + x^2 \otimes y + y \otimes x^2)$;

3) $A^7_{2|1}$ with $\Delta^7_{2|1}(x) = \frac{1}{4}(3x\otimes x + x\otimes x^2 + x^2\otimes x - x^2\otimes x^2)$,
$\Delta^7_{2|1}(y) = \frac{1}{2}y\otimes x + \frac{1}{2}x\otimes y + \frac{1}{2}x^2\otimes y + \frac{1}{2}y\otimes x^2$;

4) $A^8_{2|1}$ with $\Delta^8_{2|1}(x) = \frac{1}{4}(3x\otimes x + x\otimes x^2 + x^2\otimes x - x^2\otimes x^2)$,
$\Delta^8_{2|1}(y) = 1\otimes y + \frac{1}{2}y\otimes x + \frac{1}{2}y\otimes x^2$;

5) $A^9_{2|1}$ with $\Delta^9_{2|1}(x) = \frac{1}{2}(\frac{3}{2}x^2\otimes x^2 - \frac{1}{2}x^2\otimes x - \frac{1}{2}x\otimes x^2 + \frac{3}{2}x\otimes x + 1\otimes x - x\otimes 1 - x^2\otimes 1 - 1\otimes x^2)$,
$\Delta^9_{2|1}(y) = \frac{1}{2}(y\otimes x + x\otimes y + x^2\otimes y + y\otimes x^2)$;

6) $A^{10}_{2|1}$ with $\Delta^{10}_{2|1}(x) = \frac{1}{2}(x^2\otimes x^2 + x\otimes x + 1\otimes x + x\otimes 1 - 1\otimes x^2 - x^2\otimes 1)$,
$\Delta^{10}_{2|1}(y) = \frac{1}{2}(y\otimes x + x\otimes y + x^2\otimes y + y\otimes x^2)$;

7) $A^{11}_{2|1}$ with $\Delta^{11}_{2|1}(x) = \frac{1}{2}(x^2\otimes x^2 + x\otimes x + 1\otimes x + x\otimes 1 - 1\otimes x^2 - x^2\otimes 1)$,
$\Delta^{11}_{(2|1)}(y) = 1\otimes y + \frac{1}{2}(y\otimes x + x\otimes y + y\otimes x^2 - x^2\otimes y)$;

8) $A^{12}_{2|1}$ with $\Delta^{12}_{2|1}(x) = \frac{1}{2}(x\otimes x^2 + x\otimes x + 1\otimes x - 1\otimes x^2)$,
$\Delta^{12}_{2|1}(y) = 1\otimes y + \frac{1}{2}(y\otimes x^2 + y\otimes x)$;

9) $A^{13}_{2|1}$ with $\Delta^{13}_{2|1}(x) = -1\otimes 1 + \frac{1}{2}(x\otimes x + x\otimes 1 + 1\otimes x + 1\otimes x^2 + x^2\otimes 1 - x^2\otimes x^2)$,
$\Delta^{13}_{2|1}(y) = \frac{1}{2}(y\otimes x + x\otimes y + x^2\otimes y + y\otimes x^2)$;

10) $A^{14}_{2|1}$ with $\Delta^{14}_{2|1}(x) = 1\otimes x + x\otimes 1 - x\otimes x - x^2\otimes x - x\otimes x^2$,
$\Delta^{14}_{2|1}(y) = y\otimes 1 + 1\otimes y - x^2\otimes y - y\otimes x^2$;

11) $A^{15}_{2|1}$ with $\Delta^{15}_{2|1}(x) = 1\otimes x + x\otimes 1 - x^2\otimes x$, $\Delta^{15}_{2|1}(y) = y\otimes 1 + 1\otimes y - x^2\otimes y - y\otimes x^2$;

12) $A^{16}_{2|1}$ with $\Delta^{16}_{2|1}(x) = 1\otimes x + x\otimes 1 + \frac{1}{2}(-x\otimes x - x\otimes x^2 - x^2\otimes x + x^2\otimes x^2)$,
$\Delta^{16}_{2|1}(y) = y\otimes 1 + 1\otimes y - x^2\otimes y - y\otimes x^2$;

13) $A^{17}_{2|1}$ with $\Delta^{17}_{2|1}(x) = 1\otimes x + x\otimes 1 - x\otimes x^2 - x^2\otimes x + x^2\otimes x^2$,
$\Delta^{17}_{2|1}(y) = y\otimes 1 + 1\otimes y - x^2\otimes y - y\otimes x^2$;

14) $A^{18}_{2|1}$ with $\Delta^{18}_{2|1}(x) = -1\otimes 1 + 1\otimes x^2 + x^2\otimes 1 + \frac{1}{2}(x\otimes x + x\otimes x^2 + x^2\otimes x - 3x^2\otimes x^2)$,
$\Delta^{18}_{2|1}(y) = \frac{1}{2}y\otimes x + \frac{1}{2}x\otimes y + \frac{1}{2}x^2\otimes y + \frac{1}{2}y\otimes x^2$;

15) $A^{19}_{2|1}$ with $\Delta^{19}_{2|1}(x) = \frac{1}{2}(x\otimes x + x\otimes x^2 + x^2\otimes x - x^2\otimes x^2)$, $\Delta^{19}_{2|1}(y) = 1\otimes y + y\otimes x^2$;

16) $A^{20}_{2|1} = (A^{15}_{2|1})^{\mathrm{cop}}$, $A^{21}_{2|1} = (A^{2}_{2|1})^{\mathrm{cop}}$, $A^{22}_{2|1} = (A^{6}_{2|1})^{\mathrm{cop}}$.

Superalgebra $A_{4|1}$, we have $\Delta^k_{4|1}(x) = x\otimes x$, $\varepsilon^k_{4|1}(x) = 1$, $\varepsilon^k_{4|1}(y) = 0$ for $k = 1,\dots,3$:

1) $A^1_{4|1}$ with $\Delta^1_{4|1}(y) = x\otimes y + y\otimes x$, $\Delta^1_{4|1}(z) = x\otimes z + z\otimes x$;

2) $A^2_{4|1}$ with $\Delta^2_{4|1}(y) = x\otimes y + y\otimes 1$, $\Delta^2_{4|1}(z) = z\otimes 1 + x\otimes z$;

3) $A^3_{4|1} = (A^2_{4|1})^{\mathrm{cop}}$.

Superalgebra $A_{6|1} \cong \mathbb{K}[x,y]/(x^3 - x^2, y^2 - xy)$ with $\deg(x) = 0$ and $\deg(y) = 1$, we have $\varepsilon^k_{6|1}(x) = 1$ for $k = 1,\dots,18$:

1) $A^1_{6|1}$ with $\Delta^1_{6|1}(x) = x\otimes x^2 + x^2\otimes x - x^2\otimes x^2$, $\Delta^1_{6|1}(y) = y\otimes x^2 + x^2\otimes y$;

2) $A^2_{6|1}$ with $\Delta^2_{6|1}(x) = x\otimes x^2 + x^2\otimes x - x^2\otimes x^2 + y\otimes y$, $\Delta^2_{6|1}(y) = y\otimes x^2 + x^2\otimes y$;

3) $A^k_{6|1}$ with $\Delta^k_{6|1}(x) = x\otimes x$ for $k = 3,\dots,8$ and

- $\Delta^3_{6|1}(y) = x\otimes y + y\otimes x$,

- $\Delta^4_{6|1}(y) = y \otimes x^2 + x \otimes y$,
- $\Delta^5_{6|1}(y) = y \otimes x^2 + x^2 \otimes y$,
- $\Delta^6_{6|1}(y) = y \otimes 1 + x \otimes y$,
- $\Delta^7_{6|1}(y) = y \otimes 1 + x^2 \otimes y$,
- $\Delta^8_{6|1}(y) = 1 \otimes y + y \otimes 1$;

4) $A^9_{6|1}$ with $\Delta^9_{6|1}(x) = x \otimes x + y \otimes y$, $\Delta^9_{6|1}(y) = x \otimes y + y \otimes x$;

5) $A^{10}_{6|1}$ with $\Delta^{10}_{6|1}(x) = x \otimes 1 - x^2 \otimes 1 + x^2 \otimes x$, $\Delta^{10}_{6|1}(y) = y \otimes x^2 + x^2 \otimes y$;

6) $A^{11}_{6|1}$ with $\Delta^{11}_{6|1}(x) = x \otimes x^2 + x^2 \otimes x - x^2 \otimes x^2$, $\Delta^{11}_{6|1}(y) = y \otimes 1 + x^2 \otimes y$;

7) $A^{12}_{6|1}$ with $\Delta^{12}_{6|1}(x) = x \otimes 1 - x^2 \otimes 1 + x^2 \otimes x$, $\Delta^{12}_{6|1}(y) = x^2 \otimes y + y \otimes 1$;

8) $A^{13}_{6|1} = (A^4_{6|1})^{\mathrm{cop}}$, $A^{14}_{6|1} = (A^{11}_{6|1})^{\mathrm{cop}}$, $A^{15}_{6|1} = (A^6_{6|1})^{\mathrm{cop}}$, $A^{16}_{6|1} = (A^5_{6|1})^{\mathrm{cop}}$, $A^{17}_{6|1} = (A^{12}_{6|1})^{\mathrm{cop}}$, $A^{18}_{6|1} = (A^{10}_{6|1})^{\mathrm{cop}}$.

Superalgebra $A_{13|1} \cong \mathbb{K}\langle x, y\rangle/(x^3 - x, y^2, xy + yx, xy - y)$ with $\deg(x) = 0$ and $\deg(y) = 1$, we have $\varepsilon^k_{13|1}(x) = 0$ for $k = 1, \ldots, 11$ and $\varepsilon^k_{13|1}(x) = 1$ for $k = 12, \ldots, 21$:

1) $A^1_{13|1}$ with $\Delta^1_{13|1}(x) = 1 \otimes x + x \otimes 1 - x^2 \otimes x$, $\Delta^1_{13|1}(y) = 1 \otimes y + y \otimes 1 - x^2 \otimes y - y \otimes x^2$;

2) $A^2_{13|1}$ with $\Delta^2_{13|1}(x) = 1 \otimes x + x \otimes 1 + \frac{1}{2}x \otimes x - \frac{1}{2}x \otimes x^2 - \frac{1}{2}x^2 \otimes x - \frac{1}{2}x^2 \otimes x^2$,
$\Delta^2_{13|1}(y) = 1 \otimes y + y \otimes 1 - x^2 \otimes y - y \otimes x^2$;

3) $A^3_{13|1}$ with $\Delta^3_{13|1}(x) = 1 \otimes x + x \otimes 1 + \frac{1}{2}x \otimes x - \frac{1}{2}x \otimes x^2 - \frac{1}{2}x^2 \otimes x - \frac{1}{2}x^2 \otimes x^2$,
$\Delta^3_{13|1}(y) = 1 \otimes y + y \otimes 1$;

4) $A^4_{13|1}$ with $\Delta^4_{13|1}(x) = 1 \otimes x + x \otimes 1 + -x^2 \otimes x - x^2 \otimes x^2$,
$\Delta^4_{13|1}y) = 1 \otimes y + y \otimes 1 - y \otimes x^2 - x^2 \otimes y$;

5) $A^5_{13|1}$ with $\Delta^5_{13|1}(x) = 1 \otimes x + x \otimes 1 - \frac{1}{2}x \otimes x - \frac{1}{2}x \otimes x^2 - \frac{1}{2}x^2 \otimes x + \frac{1}{2}x^2 \otimes x^2$,
$\Delta^5_{13|1}(y) = 1 \otimes y + y \otimes 1 - \frac{1}{2}x \otimes y - \frac{1}{2}y \otimes x - \frac{1}{2}x^2 \otimes y - \frac{1}{2}y \otimes x^2$;

6) $A^6_{13|1}$ with $\Delta^6_{13|1}(x) = 1 \otimes x + x \otimes 1 - \frac{1}{2}x \otimes x - \frac{1}{2}x \otimes x^2 - \frac{1}{2}x^2 \otimes x + \frac{1}{2}x^2 \otimes x^2 - 2y \otimes y$,
$\Delta^6_{13|1}(y) = 1 \otimes y + y \otimes 1 - \frac{1}{2}x \otimes y - \frac{1}{2}y \otimes x - \frac{1}{2}x^2 \otimes y - \frac{1}{2}y \otimes x^2$;

7) $A^7_{13|1}$ with $\Delta^7_{13|1}(x) = 1 \otimes x + x \otimes 1 - \frac{1}{2}x \otimes x - \frac{1}{2}x \otimes x^2 + \frac{1}{2}x^2 \otimes x^2 - x^2 \otimes x$,
$\Delta^7_{13|1}(y) = 1 \otimes y + y \otimes 1 - \frac{1}{2}x \otimes y - \frac{1}{2}x^2 \otimes y - y \otimes x^2$;

8) $A^8_{13|1}$ with $\Delta^8_{13|1}(x) = 1 \otimes x + x \otimes 1 - x \otimes x^2$, $\Delta^8_{13|1}(y) = 1 \otimes y + y \otimes 1 - x^2 \otimes y - y \otimes x^2$;

9) $A^9_{13|1}$ with $\Delta^9_{13|1}(x) = 1 \otimes x + x \otimes 1 - x \otimes x^2$, $\Delta^9_{13|1}(y) = y \otimes 1 + 1 \otimes y - y \otimes x^2$;

10) $A^{10}_{13|1} = (A^9_{13|1})^{\mathrm{cop}}$;

11) $A^{11}_{13|1}$ with $\Delta^{11}_{13|1}(x) = 1 \otimes x + x \otimes 1 - x \otimes x - x \otimes x^2 - x^2 \otimes x$,
$\Delta^{11}_{13|1}(y) = 1 \otimes y + y \otimes 1 - x^2 \otimes y - y \otimes x^2$;

12) $A^{12}_{13|1}$ with $\Delta^{12}_{13|1}(x) = \frac{1}{2}1 \otimes x + \frac{1}{2}x \otimes 1 - \frac{1}{2}1 \otimes x^2 - \frac{1}{2}x^2 \otimes 1 - \frac{1}{4}x \otimes x^2 - \frac{1}{4}x^2 \otimes x + \frac{3}{4}x \otimes x + \frac{3}{4}x^2 \otimes x^2$,
$\Delta^{12}_{13|1}(y) = \frac{1}{2}x \otimes y + \frac{1}{2}y \otimes x + \frac{1}{2}x^2 \otimes y + \frac{1}{2}y \otimes x^2$;

13) $A^k_{13|1}$ with $\Delta^k_{13|1}(y) = \frac{1}{2}x \otimes y + \frac{1}{2}y \otimes x + \frac{1}{2}x^2 \otimes y + \frac{1}{2}y \otimes x^2$ for $k = 13, \ldots, 21$:

- $\Delta^{13}_{13|1}(x) = x \otimes x$,
- $\Delta^{14}_{13|1}(x) = \frac{1}{2}x \otimes x + \frac{1}{2}x \otimes x^2 + \frac{1}{2}x^2 \otimes x - \frac{1}{2}x^2 \otimes x^2 + 2y \otimes y$,

- $\Delta^{15}_{13|1}(x) = \frac{1}{2}x\otimes x + \frac{1}{2}x^2\otimes x + \frac{1}{2}x\otimes 1 - \frac{1}{2}x^2\otimes 1$,
- $\Delta^{16}_{13|1}(x) = \frac{1}{2}1\otimes x + \frac{1}{2}x\otimes 1 - \frac{1}{2}1\otimes x^2 - \frac{1}{2}x^2\otimes 1 + \frac{1}{2}x\otimes x + \frac{1}{2}x^2\otimes x^2$,
- $\Delta^{17}_{13|1}(x) = \frac{3}{4}x\otimes x + \frac{1}{4}x\otimes x^2 + \frac{1}{4}x^2\otimes x - \frac{1}{4}x^2\otimes x^2$,
- $\Delta^{18}_{13|1}(x) = -1\otimes 1 + \frac{1}{2}x\otimes x + 1\otimes x^2 + x^2\otimes 1 + \frac{1}{2}x\otimes x^2 + \frac{1}{2}x^2\otimes x - \frac{3}{2}x^2\otimes x^2$,
- $\Delta^{19}_{13|1}(x) = -1\otimes 1 + \frac{1}{2}x\otimes x + 1\otimes x^2 + x^2\otimes 1 + \frac{1}{2}x\otimes x^2 + \frac{1}{2}x^2\otimes x - \frac{3}{2}x^2\otimes x^2 + 2y\otimes y$,
- $\Delta^{20}_{13|1}(x) = -1\otimes 1 + \frac{1}{2}x\otimes x + \frac{1}{2}1\otimes x^2 + \frac{1}{2}x^2\otimes 1 + \frac{1}{2}x\otimes x^2 + \frac{1}{2}x^2\otimes x - \frac{3}{2}x^2\otimes x^2$;

14) $A^{21}_{13|1} = (A^{15}_{13|1})^{\mathrm{cop}}$.

Superalgebra $A_{14|1}$, we have $\varepsilon^k_{14|1}(x) = 0$, $\varepsilon^k_{14|1}(y) = 0$ for $k = 1, \ldots, 9$:

1) $A^k_{14|1}$ with $\Delta^1_{14|1}(x) = 1\otimes x + x\otimes 1 - x\otimes x$ for $k = 1, \ldots, 6$:

- $\Delta^1_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x$, $\Delta^1_{14|1}(z) = 1\otimes z + z\otimes 1$,
- $\Delta^2_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x$, $\Delta^2_{14|1}(z) = 1\otimes z + z\otimes 1 - x\otimes z$,
- $\Delta^3_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x + y\otimes y$, $\Delta^3_{14|1}(z) = 1\otimes z + z\otimes 1 - x\otimes z$,
- $\Delta^4_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x$, $\Delta^4_{14|1}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x$,
- $\Delta^5_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x + y\otimes y$, $\Delta^5_{14|1}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x + z\otimes y$,
- $\Delta^6_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes z - z\otimes x + y\otimes y$, $\Delta^6_{14|1}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x$;

2) $A^7_{14|1}$ with $\Delta^7_{14|1}(x) = 1\otimes x + x\otimes 1 - x\otimes x + y\otimes y$,
$\Delta^7_{14|1}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x + y\otimes y$, $\Delta^7_{14|1}(z) = 1\otimes z + z\otimes 1$;

3) $A^8_{14|1} = (A^2_{14|1})^{\mathrm{cop}}$, $A^9_{14|1} = (A^3_{14|1})^{\mathrm{cop}}$.

Superalgebra $A_{14|2}$, we have $\varepsilon^k_{14|2}(x) = 0$ and $\varepsilon^k_{14|2}(y) = 0$ for $k = 1, \ldots, 4$:

1) $A^1_{14|2}$ with $\Delta^1_{14|2}(x) = 1\otimes x + x\otimes 1 - x\otimes x$, $\Delta^1_{14|2}(y) = 1\otimes y + y\otimes 1 - y\otimes z - z\otimes x$,
$\Delta^1_{14|2}(z) = 1\otimes z + z\otimes 1 - z\otimes x - x\otimes z$;

2) $A^2_{14|2}$ with $\Delta^2_{14|2}(x) = 1\otimes x + x\otimes 1 - x\otimes x$, $\Delta^2_{14|2}(y) = 1\otimes y + y\otimes 1 - x\otimes y$,
$\Delta^2_{14|2}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x$;

3) $A^3_{14|2}$ with $\Delta^3_{14|2}(x) = 1\otimes x + x\otimes 1 - x\otimes x$, $\Delta^3_{14|2}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x + y\otimes y$,
$\Delta^3_{14|2}(z) = 1\otimes z + z\otimes 1 - z\otimes x - x\otimes z$;

4) $A^4_{14|2} = (A^2_{14|2})^{\mathrm{cop}}$.

Superalgebra $A_{15|1}$, we have $A^1_{15|1} = (A^1_{14|1})^{\mathrm{op}}$, $A^2_{15|1} = (A^2_{14|1})^{\mathrm{op}}$, $A^3_{15|1} = (A^5_{14|1})^{\mathrm{op}}$, $A^4_{15|1} = (A^9_{14|1})^{\mathrm{op}}$, $A^5_{15|1} = (A^6_{14|1})^{\mathrm{op}}$, $A^6_{15|1} = (A^3_{14|1})^{\mathrm{op}}$, $A^7_{15|1} = (A^4_{14|1})^{\mathrm{op}}$, $A^8_{15|1} = (A^7_{14|1})^{\mathrm{op}}$, $A^9_{15|1} = (A^4_{14|1})^{\mathrm{op,cop}}$.

Superalgebra $A_{15|2}$, we have $A^1_{15|2} = (A^1_{14|2})^{\mathrm{op}}$, $A^2_{15|2} = (A^4_{14|2})^{\mathrm{op}}$, $A^3_{15|2} = (A^2_{14|2})^{\mathrm{op}}$, $A^4_{15|2} = (A^3_{14|2})^{\mathrm{op}}$.

Superalgebra $A_{17|1}$, we have $\varepsilon^k_{17|1}(x) = 1$ and $\varepsilon^k_{17|1}(y) = 0$ for $k = 1, \ldots, 11$:

1) $A^1_{17|1}$ with $\Delta^1_{17|1}(x) = x\otimes x + y\otimes y + z\otimes z$, $\Delta^1_{17|1}(y) = x\otimes y + y\otimes x + z\otimes z$,
$\Delta^1_{17|1}(z) = x\otimes z + z\otimes x + z\otimes y + y\otimes z$;

2) $A^2_{17|1}$ with $\Delta^2_{17|1}(x) = x\otimes x + z\otimes z$, $\Delta^2_{17|1}(y) = x\otimes y + y\otimes x$, $\Delta^2_{17|1}(z) = z\otimes x + x\otimes z$;

3) $A^3_{17|1}$ with $\Delta^3_{17|1}(x) = x\otimes x + z\otimes z$, $\Delta^3_{17|1}(y) = x\otimes y + y\otimes x + z\otimes z - y\otimes y$, $\Delta^3_{17|1}(z) = z\otimes x + x\otimes z$;

4) $A^4_{17|1}$ with $\Delta^4_{17|1}(x) = x\otimes x + y\otimes y$, $\Delta^4_{17|1}(y) = x\otimes y + y\otimes x$, $\Delta^4_{17|1}(z) = z\otimes x + x\otimes z - z\otimes y - y\otimes z$;

5) $A^k_{17|1}$, we have $\Delta^k_{17|1}(x) = x\otimes x$ and $\Delta^k_{17|1}(y) = x\otimes y + y\otimes x + y\otimes y$ for $k = 5, 6, 7$:

- $\Delta^5_{17|1}(z) = x\otimes z + z\otimes x$,
- $\Delta^6_{17|1}(z) = x\otimes z + z\otimes x + z\otimes y - y\otimes z$,
- $\Delta^7_{17|1}(z) = z\otimes x + x\otimes z + z\otimes y$;

6) $A^8_{17|1} = (A^7_{17|1})^{\mathrm{cop}}$;

7) $A^k_{17|1}$, we have $\Delta^k_{17|1}(x) = x\otimes x$ for $k = 9, 10, 11$:

- $\Delta^9_{17|1}(y) = x\otimes y + y\otimes x - y\otimes y$, $\Delta^9_{17|1}(z) = z\otimes x + x\otimes z$,
- $\Delta^{10}_{17|1}(y) = x\otimes y + y\otimes x + z\otimes z$, $\Delta^{10}_{17|1}(z) = x\otimes z + z\otimes x$,
- $\Delta^{11}_{17|1}(y) = x\otimes y + y\otimes x + y\otimes y + z\otimes z$, $\Delta^{11}_{17|1}(z) = z\otimes x + x\otimes z + z\otimes y + y\otimes z$.

A.2 Case dim $A_0 = 2$

Superalgebra $A_{2|3}$, we have $\Delta^k_{2|3}(x) = 1\otimes x + x\otimes 1 - x\otimes x$, $\varepsilon^k_{2|3}(x) = 0$ for $k = 1, \ldots, 4$:

1) $A^1_{2|3}$ with $\Delta^1_{2|3}(y) = 1\otimes y + y\otimes 1 - x\otimes y$, $\Delta^1_{2|3}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x$;

2) $A^2_{2|3}$ with $\Delta^2_{2|3}(y) = 1\otimes y + y\otimes 1 - x\otimes y + z\otimes x$, $\Delta^2_{2|3}(z) = 1\otimes z + z\otimes 1 - x\otimes z - z\otimes x$;

3) $A^3_{2|3} = (A^1_{2|3})^{\mathrm{cop}}$, $A^4_{2|3} = (A^2_{2|3})^{\mathrm{cop}}$.

Superalgebra $A_{3|2} \cong \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, yx)$ with $\deg(x) = 0$ and $\deg(y) = 1$, we have $\varepsilon^k_{3|2}(x) = 0$ for $k = 1, \ldots, 9$:

1) $A^k_{3|2}$ with $\Delta^k_{3|2}(x) = 1\otimes x + x\otimes 1 - 2x\otimes x$ for $k = 1, 2$, and

- $\Delta^1_{3|2}(y) = 1\otimes y + y\otimes 1 - 2x\otimes y$,
- $\Delta^2_{3|2}(y) = 1\otimes y + y\otimes 1$;

2) $A^k_{3|2}$ with $\Delta^k_{3|2}(x) = 1\otimes x + x\otimes 1 - x\otimes x$, for $k = 3, \ldots, 9$, and

- $\Delta^3_{3|2}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x + xy\otimes x + x\otimes xy$,
- $\Delta^4_{3|2}(y) = 1\otimes y + y\otimes 1 - x\otimes y - xy\otimes x$,
- $\Delta^5_{3|2}(y) = 1\otimes y + y\otimes 1 - x\otimes y - y\otimes x$,
- $\Delta^6_{3|2}(y) = 1\otimes y + y\otimes 1 - x\otimes xy - xy\otimes x$,
- $\Delta^7_{3|2}(y) = 1\otimes y + y\otimes 1 - y\otimes x$,
- $\Delta^8_{3|2}(y) = 1\otimes y + y\otimes 1$,
- $A^9_{3|2} = (A^4_{3|2})^{\mathrm{cop}}$.

Superalgebra $A_{6|2}$, we have $\Delta^k_{6|2}(x) = x\otimes x$ and $\varepsilon^k_{6|2}(x) = 1$ for $k = 1, \ldots, 11$:

1) $A^k_{6|2}$ with $\Delta^k_{6|2}(y) = y\otimes x + x\otimes y$ for $k = 1, \ldots, 4$, and

- $\Delta^1_{6|2}(z) = z \otimes 1 + 1 \otimes z$,
- $\Delta^2_{6|2}(z) = z \otimes x + x \otimes z$,
- $\Delta^3_{6|2}(z) = z \otimes x + 1 \otimes z$,
- $\Delta^4_{6|2}(z) = 1 \otimes z + z \otimes x + 1 \otimes y - x \otimes y$;

2) $A^5_{6|2}$ with $\Delta^6_{6|2}(y) = y \otimes 1 + 1 \otimes y$, $\Delta^5_{6|2}(z) = z \otimes x + x \otimes z$;

3) $A^6_{6|2}$ with $\Delta^6_{6|2}(y) = y \otimes x + 1 \otimes y$, $\Delta^6_{6|2}(z) = z \otimes x + 1 \otimes z$;

4) $A^7_{6|2}$ with $\Delta^7_{6|2}(y) = 1 \otimes y + y \otimes x$, $\Delta^7_{6|2}(z) = x \otimes z + z \otimes x$;

5) $A^8_{6|2}$ with $\Delta^8_{6|2}(y) = y \otimes x + x \otimes y + z \otimes 1 + 1 \otimes z - x \otimes z - z \otimes x$, $\Delta^8_{6|2}(z) = 1 \otimes z + z \otimes 1$;

6) $A^9_{6|2}$ with $\Delta^9_{6|2}(y) = y \otimes x + 1 \otimes y + z \otimes 1 + 1 \otimes z - z \otimes x - x \otimes z$, $\Delta^9_{6|2}(z) = z \otimes x + 1 \otimes z$;

7) $A^{10}_{6|2}$ with $\Delta^{10}_{6|2}(y) = y \otimes x + \frac{1}{2}x \otimes y + \frac{1}{2}1 \otimes y + \frac{1}{4}1 \otimes z - \frac{1}{4}x \otimes z$,
$\Delta^{10}_{6|2}(z) = z \otimes x + \frac{1}{2}1 \otimes z + 1 \otimes y - x \otimes y + \frac{1}{2}x \otimes z$;

8) $A^{11}_{6|2} = (A^6_{3|2})^{\mathrm{cop}}$.

Superalgebra $A_{11|2} \cong \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, xy - yx - y)$ with $\deg(x) = 0, \deg(y) = 1$:

1) $\Delta^1_{11|2}(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x$, $\Delta^1_{11|2}(y) = 1 \otimes y + y \otimes 1 - 2x \otimes xy - 2xy \otimes x$,
$\varepsilon^1_{11|2}(x) = 1$.

Superalgebra $A^1_{12|2} \cong \mathbb{K}\langle x, y\rangle/(x^2, y^2, xy + yx)$ with $\deg(x) = \deg(y) = 1$:

1) $\Delta^1_{12|2}(x) = 1 \otimes x + x \otimes 1$, $\Delta^1_{12|2}(y) = 1 \otimes y + y \otimes 1$.

Superalgebra $A_{14|3}$, we have $\Delta^k_{14|3}(z) = 1 \otimes z + z \otimes 1 - x \otimes z - z \otimes x$ and $\varepsilon^k_{14|3}(x) = 0$ for $k = 1, \ldots, 7$:

1) $A^k_{14|3}$ with $\Delta^k_{14|3}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$ for $k = 1, \ldots, 4$:

- $\Delta^1_{14|3}(y) = 1 \otimes y + y \otimes 1$,
- $\Delta^2_{14|3}(y) = 1 \otimes y + y \otimes 1 + x \otimes z + z \otimes x$,
- $\Delta^3_{14|3}(y) = 1 \otimes y + y \otimes 1 - x \otimes y$,
- $\Delta^4_{14|3}(y) = 1 \otimes y + y \otimes 1 - x \otimes y - y \otimes x$;

2) $A^k_{14|3}$ with $\Delta^k_{14|3}(x) = 1 \otimes x + x \otimes 1 - x \otimes x + z \otimes z$ for $k = 5, 6$:

- $\Delta^5_{14|3}(y) = 1 \otimes y + y \otimes 1$,
- $\Delta^6_{14|3}(y) = 1 \otimes y + y \otimes 1 + x \otimes z + z \otimes x$;

3) $A^7_{14|3} = (A^3_{14|3})^{\mathrm{cop}}$.

Superalgebra $A_{15|3}$, we have $A^1_{15|3} = (A^1_{14|3})^{\mathrm{op}}$, $A^2_{15|3} = (A^2_{14|3})^{\mathrm{op}}$, $A^3_{15|3} = (A^3_{14|3})^{\mathrm{op}}$, $A^4_{15|3} = (A^4_{14|3})^{\mathrm{op}}$, $A^5_{15|3} = (A^5_{14|3})^{\mathrm{op}}$, $A^6_{15|3} = (A^6_{14|3})^{\mathrm{op}}$, $A^7_{15|3} = (A^5_{14|3})^{\mathrm{op,cop}}$.
Superalgebra $A_{17|2}$, we have:

1) $A^1_{17|2}$ with $\Delta^1_{17|2}(x) = 1 \otimes x + x \otimes 1 - x \otimes x$, $\Delta^1_{17|2}(y) = 1 \otimes y + y \otimes 1 - y \otimes x - x \otimes y$,
$\Delta^1_{17|2}(z) = 1 \otimes z + z \otimes 1 - z \otimes x - x \otimes z$, $\varepsilon^7_{15|3}(x) = 0$;

2) $A^2_{17|2}$ with $\Delta^2_{17|2}(x) = x \otimes x$, $\Delta^2_{17|2}(y) = x \otimes y + y \otimes x$, $\Delta^2_{17|2}(z) = x \otimes z + z \otimes x$, $\varepsilon^7_{15|3}(x) = 1$.

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